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Fusion procedure for open chains

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Abstract. We have recently generalized Sklyanin's approach of constructing open integrable quantum spin chains to the case of PT-invariant R matrices. Here we formulate a fusion procedure for such chains. In particular, we show that the fused transfer matrix can be expressed in terms of products of the original transfer matrix and products of certain quantum determinants which can be explicitly evaluated. Applications of these results include constructing open integrable higher-spin chains, as well as obtaining functional equations for transfer-matrix eigenvalues, which may be solved by an analytical Bethe ansatz.

1. Introduction

Given a trigonometric R matrix (i.e. a solution of the Yang–Baxter equation) related to the fundamental representation of a Lie algebra g, the so-called fusion procedure [1] provides a way of constructing new R matrices related to higher-dimensional representations of this algebra. Such an R matrix can be used to construct a closed integrable quantum spin chain of higher spin, whose transfer matrix is related to that of the corresponding chain in the fundamental representation. That is, the fusion procedure for R matrices implies a fusion procedure for the corresponding closed-chain transfer matrices. This result is important for solving [1, 2] the higher-spin chain. Moreover, this result also leads to a functional equation for the eigenvalues of these transfer matrices. This functional equation, together with an additional equation implied by the unitarity of the R matrix, can be solved through the so-called analytical [3] Bethe ansatz.

Sklyanin [4] introduced a novel approach for constructing *open* integrable quantum spin chains, whose R matrices are symmetric (i.e. both P and T invariant) as well as unitary and crossing-unitary. An important element of this approach is the so-called K matrix, which can be interpreted as the amplitude for a particle to reflect elastically from a wall. In general, there is a family of K matrices which is compatible with a given R matrix. Given such R and K matrices, one can construct a corresponding open-chain transfer matrix. For the special case of spin-$\frac{1}{2}$ $A_1^{(1)}$, the eigenstates and eigenvalues of this transfer matrix can be determined by an algebraic Bethe ansatz. (This model was first solved via the coordinate Bethe ansatz by Alcaraz *et al* [5].)

In [6], a fusion procedure for $A_1^{(1)}$ K matrices was formulated, and was used to solve the open spin-1 $A_1^{(1)}$ chain with boundary terms.

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For particular choices of boundary terms (corresponding to particular K matrices), the spin $s = \frac{1}{2}$ and 1 $A_1^{(1)}$ chains have the quantum algebra symmetry $U_q[\mathrm{su}(2)]$ [7–9]. For q a primitive root of unity, these chains are related to the $c < 1$ and $c < \frac{3}{2}$ unitary minimal models, respectively [5, 7].

We have recently generalized [10] Sklyanin's approach of constructing open integrable chains to include non-symmetric R matrices which are only PT invariant. All of the trigonometric R matrices found by Bazhanov [11] and Jimbo [12] are of this type. We have used this formalism to construct a large class of integrable models which have quantum algebra invariance [13, 14].

In this paper, we formulate a fusion procedure for K matrices corresponding to such non-symmetric R matrices. Furthermore, we deduce a fusion procedure for the corresponding open-chain transfer matrices. These results can be used to construct integrable open chains with spins in higher-dimensional representations of an arbitrary algebra g. In general, these representations are reducible, i.e. the chains involve spins in two or more irreducible representations. This suggests that there may be a further higher symmetry in such models. (The same is true for the corresponding closed chains.)

In a separate publication [15], we use these fusion results and an analytical Bethe ansatz to solve the large class of models found in [13] and [14] which have quantum algebra symmetry. We find that the Bethe ansatz equations for these open chains are given by certain 'doublings' of the Bethe ansatz equations for the corresponding closed chains.

2. Fusion procedure for R matrices

We begin by reviewing the fusion procedure for R matrices. We assume that the R matrix, which acts on $C^n \otimes C^n$, obeys the Yang–Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v) \tag{2.1}$$

as well as PT symmetry

$$\mathcal{P}_{12}R_{12}(u)\mathcal{P}_{12} \equiv R_{21}(u) = R_{12}(u)^{t_1 t_2} \tag{2.2}$$

where $\mathcal{P}_{12}$ is the permutation matrix, $\mathcal{P}_{12}(x \otimes y) = y \otimes x$ for $x, y \in C^n$, and t_i denotes transposition in the ith space; unitarity

$$R_{12}(u)R_{21}(-u) = \zeta(u) \tag{2.3}$$

where $\zeta(u)$ is some even scalar function of u; crossing symmetry

$$R_{12}(u) = V_1 R_{12}(-u-\rho)^{t_2} V_1 = V_2^{t_2} R_{12}(-u-\rho)^{t_1} V_2^{t_2} \tag{2.4}$$

where $V^2 = 1$ (in this paper, we use the notation $V_1 = V \otimes 1$, $V_2 = 1 \otimes V$); and regularity

$$R_{12}(0) = \zeta(0)^{1/2}\mathcal{P}_{12}. \tag{2.5}$$

We record here for future reference the useful identity

$$V_1 R_{12}(u) V_1 = V_2 R_{21}(u) V_2. \tag{2.6}$$

A fact which is essential for establishing the fusion procedure is that at $u = -\rho$, the matrix $R_{12}(u)$ degenerates to a quantity proportional to a projector onto a one-dimensional subspace. Indeed, one can show that

$$\tilde{P}^-_{12} = \frac{1}{n\zeta(0)^{1/2}} R_{12}(-\rho) = \frac{1}{n} V_1 \mathcal{P}^{t_2}_{12} V_1 \tag{2.7}$$

obeys

$$\left(\tilde{P}^-_{12}\right)^2 = \tilde{P}^-_{12} \tag{2.8}$$

and

$$\tilde{P}^-_{12} A_{12} \tilde{P}^-_{12} = \mathrm{tr}_{12}\left(\tilde{P}^-_{12} A_{12}\right) \tilde{P}^-_{12} \tag{2.9}$$

where A is an arbitrary matrix acting on $C^n \otimes C^n$. Thus, $\tilde{P}^+_{12}$ defined as

$$\tilde{P}^+_{12} = 1 - \tilde{P}^-_{12} \tag{2.10}$$

is also a projector. Note that these projectors are not symmetric, $(\tilde{P}^{\mp}_{12})^{t_1 t_2} = \tilde{P}^{\mp}_{21} \neq \tilde{P}^{\mp}_{12}$.

For $v = u + \rho$, the Yang–Baxter equation (2.1) degenerates to

$$\tilde{P}^-_{12} R_{13}(u) R_{23}(u+\rho) = R_{23}(u+\rho) R_{13}(u) \tilde{P}^-_{12} \tag{2.11}$$

which implies that

$$\tilde{P}^-_{12} R_{13}(u) R_{23}(u+\rho) \tilde{P}^+_{12} = 0. \tag{2.12}$$

This result can be used to show that the 'fused' R matrix

$$R_{\{12\}3}(u) = \tilde{P}^+_{12} R_{13}(u) R_{23}(u+\rho) \tilde{P}^+_{12} \tag{2.13}$$

obeys a generalized Yang–Baxter equation,

$$R_{\{12\}3}(u-v) R_{\{12\}4}(u) R_{34}(v) = R_{34}(v) R_{\{12\}4}(u) R_{\{12\}3}(u-v). \tag{2.14}$$

Because the projectors $\tilde{P}^{\mp}_{12}$ are not symmetric, these equations have a second solution,

$$R'_{\{12\}3}(u) = R_{\{21\}3}(u-\rho) = \tilde{P}^+_{21} R_{23}(u-\rho) R_{13}(u) \tilde{P}^+_{21}. \tag{2.15}$$

These two solutions are presumably related by a gauge transformation.

Similarly, one finds the fused R matrices

$$R_{3\{12\}}(u) = \tilde{P}^+_{12} R_{32}(u-\rho) R_{31}(u) \tilde{P}^+_{12} \tag{2.16}$$

$$R'_{3\{12\}}(u) = R_{3\{21\}}(u+\rho) = \tilde{P}^+_{21} R_{31}(u) R_{32}(u+\rho) \tilde{P}^+_{21}. \tag{2.17}$$

Indeed, one can generate an entire hierarchy of fused R matrices which obey Yang–Baxter equations of the form

$$R_{s_1 s_2}(u-v)R_{s_1 s_3}(u)R_{s_2 s_3}(v) = R_{s_2 s_3}(v)R_{s_1 s_3}(u)R_{s_1 s_2}(u-v). \tag{2.18}$$

However, we shall not explicitly consider higher fused R matrices here.

The unitarity and crossing-symmetry properties of the fused R matrices can be determined from a knowledge of the corresponding properties of the original R matrix. We find that

$$R_{\langle 12\rangle 3}(u)R_{3\langle 12\rangle}(-u) = \zeta(u)\zeta(u+\rho)\tilde{P}^{+}_{12} \tag{2.19}$$

and

$$R_{\langle 12\rangle 3}(u) = V_{\langle 21\rangle}R'_{\langle 12\rangle 3}(-u-\rho)^{t_3}V_{\langle 12\rangle} = V_3^{t_3}R_{3\langle 12\rangle}(-u-\rho)^{t_3}V_3^{t_3} \tag{2.20}$$

$$R_{3\langle 12\rangle}(u) = V_3 R_{\langle 12\rangle 3}(-u-\rho)^{t_3}V_3 = V_{\langle 21\rangle}^{t_{12}}R'_{3\langle 12\rangle}(-u-\rho)^{t_3}V_{\langle 12\rangle}^{t_{12}} \tag{2.21}$$

where

$$V_{\langle 12\rangle} = \tilde{P}^{+}_{21}V_1V_2\tilde{P}^{+}_{12}. \tag{2.22}$$

In obtaining the crossing-symmetry formulae, we use the result

$$\tilde{P}^{-}_{21}V_1V_2\tilde{P}^{+}_{12} = 0 \tag{2.23}$$

which follows from the degeneration of the identity (2.6) at $u = -\rho$.

3. Fusion procedure for K matrices

As mentioned in the introduction, K matrices are important elements in Sklyanin's construction of open integrable chains. For a PT-invariant R matrix, the fundamental 'reflection–factorization' relations obeyed by $K^-(u)$ and $K^+(u)$ are [4, 10]

$$R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v) \tag{3.1}$$

and

$$\begin{aligned} &R_{12}(-u+v)K_1^+(u)^{t_1}M_1^{-1}R_{21}(-u-v-2\rho)M_1K_2^+(v)^{t_2} \\ &\quad = K_2^+(v)^{t_2}M_1R_{12}(-u-v-2\rho)M_1^{-1}K_1^+(u)^{t_1}R_{21}(-u+v) \end{aligned} \tag{3.2}$$

respectively, where $M \equiv V^tV$. These relations correspond to the constraint of factorized scattering in the presence of a wall.

In order to obtain a fusion formula for $K^-(u)$, we follow the same strategy employed for R matrices: namely, we consider the degeneration of its defining relation. Setting $v = u+\rho$ in (3.1), we obtain

$$\tilde{P}^{-}_{12}K_1^-(u)R_{21}(2u+\rho)K_2^-(u+\rho) = K_2^-(u+\rho)R_{12}(2u+\rho)K_1^-(u)\tilde{P}^{-}_{21} \tag{3.3}$$

which implies that

$$\tilde{P}^-_{12} K^-_1(u) R_{21}(2u+\rho) K^-_2(u+\rho) \tilde{P}^+_{21} = 0. \tag{3.4}$$

This result can be used to show that the fused K^- matrix

$$K^-_{\langle 12 \rangle}(u) = \tilde{P}^+_{12} K^-_1(u) R_{21}(2u+\rho) K^-_2(u+\rho) \tilde{P}^+_{21} \tag{3.5}$$

obeys the reflection–factorization relation

$$R_{3\langle 12 \rangle}(u-v) K^-_3(u) R_{\langle 12 \rangle 3}(u+v) K^-_{\langle 12 \rangle}(v)$$
$$= K^-_{\langle 12 \rangle}(v) R'_{3\langle 12 \rangle}(u+v) K^-_3(u) R'_{\langle 12 \rangle 3}(u-v). \tag{3.6}$$

The quantity $K'^-_{\langle 12 \rangle}(u) = K^-_{\langle 21 \rangle}(u-\rho)$ satisfies a similar relation, with R and R' interchanged.

We now turn to $K^+(u)$. The degeneration of (3.2) implies (after permuting $1 \leftrightarrow 2$, transposing, and shifting $u \to u+\rho$) that

$$\tilde{P}^+_{21} K^+_1(u)^{t_1} M_2 R_{21}(-2u-3\rho) M_2^{-1} K^+_2(u+\rho)^{t_2} \tilde{P}^-_{12} = 0. \tag{3.7}$$

This result and the identity

$$M_1^{-1} R_{12}(u) M_1 = M_2 R_{12}(u) M_2^{-1} \tag{3.8}$$

can be used to show that the fused K^+ matrix

$$K^+_{\langle 12 \rangle}(u)^{t_{12}} = \tilde{P}^+_{21} K^+_1(u)^{t_1} M_2 R_{21}(-2u-3\rho) M_2^{-1} K^+_2(u+\rho)^{t_2} \tilde{P}^+_{12} \tag{3.9}$$

obeys the relation

$$R_{\langle 12 \rangle 3}(-u+v)^{t_{123}} K^+_3(u)^{t_3} M_3^{-1} R_{3\langle 12 \rangle}(-u-v-2\rho)^{t_{123}} M_3 K^+_{\langle 12 \rangle}(v)^{t_{12}}$$
$$= K^+_{\langle 12 \rangle}(v)^{t_{12}} M_3 R'_{\langle 12 \rangle 3}(-u-v-2\rho)^{t_{123}} M_3^{-1} K^+_3(u)^{t_3} R'_{3\langle 12 \rangle}(-u+v)^{t_{123}}. \tag{3.10}$$

Further justification for considering this relation will be given in the next section.

There is an automorphism [4, 10] between the K^- and K^+ relations, (3.1) and (3.2): namely, given a solution $K^-(u)$, then $K^-(-u-\rho)^t M$ is a solution $K^+(u)$. One can establish a similar automorphism between the relation for $K^+_{\langle 12 \rangle}(u)$ (3.10) and the relation for $K'^-_{\langle 12 \rangle}(u)$.

4. Fusion procedure for transfer matrices

Given an R matrix with the properties (2.1)–(2.5) and $K^{\mp}$ matrices satisfying (3.1) and (3.2), the corresponding open-chain transfer matrix $t(u)$ is given by [4]

$$t(u) = \mathrm{tr}_1\, K_1^+(u)\mathcal{T}_1^-(u) \tag{4.1}$$

where $\mathcal{T}^-(u)$ is given by

$$\mathcal{T}^-(u) = T(u)K^-(u)\hat{T}(u) \tag{4.2}$$

and $T(u)$, $\hat{T}(u)$ obey

$$\begin{aligned} R_{12}(u-v)T_1(u)T_2(v) &= T_2(v)T_1(u)R_{12}(u-v) \\ \hat{T}_2(v)R_{12}(u+v)T_1(u) &= T_1(u)R_{12}(u+v)\hat{T}_2(v) \\ R_{12}(-u+v)\hat{T}_2(v)\hat{T}_1(u) &= \hat{T}_1(u)\hat{T}_2(v)R_{12}(-u+v). \end{aligned} \tag{4.3}$$

$T(u)$ is called the monodromy matrix, and $\hat{T}(u)$ obeys the same algebraic relations as $T(-u)^{-1}$. The quantity $\mathcal{T}^-(u)$ satisfies the same relation (3.1) as $K^-(u)$; and the transfer matrix constitutes a one-parameter commutative family

$$[t(u), t(v)] = 0 \qquad \text{for all } u, v. \tag{4.4}$$

Our first task is to construct a transfer matrix $\tilde{t}(u)$ for the fused quantities that we have described. Let us consider

$$\tilde{t}(u) = \mathrm{tr}_{12}\, K^+_{\langle 12\rangle}(u)\mathcal{T}^-_{\langle 12\rangle}(u) \tag{4.5}$$

where

$$\mathcal{T}^-_{\langle 12\rangle}(u) = T_{\langle 12\rangle}(u)K^-_{\langle 12\rangle}(u)\hat{T}_{\langle 21\rangle}(u+\rho) \tag{4.6}$$

and

$$T_{\langle 12\rangle}(u) = \tilde{P}^+_{12}T_1(u)T_2(u+\rho)\tilde{P}^+_{12} \qquad \hat{T}_{\langle 21\rangle}(u+\rho) = \tilde{P}^+_{21}\hat{T}_1(u)\hat{T}_2(u+\rho)\tilde{P}^+_{21}. \tag{4.7}$$

We observe that $\mathcal{T}^-_{\langle 12\rangle}(u)$ obeys the same relation (3.6) as $K^-_{\langle 12\rangle}(u)$ and, therefore, the same fusion formula holds:

$$\mathcal{T}^-_{\langle 12\rangle}(u) = \tilde{P}^+_{12}\mathcal{T}_1^-(u)R_{21}(2u+\rho)\mathcal{T}_2^-(u+\rho)\tilde{P}^+_{21}. \tag{4.8}$$

In order to establish the commutativity

$$[t(u), \tilde{t}(v)] = 0 \tag{4.9}$$

we generalize a similar computation described by Sklyanin [4]. In particular, we make use of the unitarity relation (2.19), as well as the relation

$$M_3^{-1} R_{\langle 12\rangle 3}(-u-2\rho)^{t_{12}} M_3 R_{3\langle 12\rangle}(u)^{t_{12}} = \zeta(u)\zeta(u+\rho)\tilde{P}_{12}^{+} \tag{4.10}$$

which follows from (2.19)–(2.21). Moreover, we also use the reflection–factorization relations (3.6) and (3.10) which are obeyed by $K_{\langle 12\rangle}^{-}$ and $K_{\langle 12\rangle}^{+}$, respectively. Indeed, starting from the $K_{\langle 12\rangle}^{-}$ relation, it was by demanding the commutativity (4.9) that we first obtained the $K_{\langle 12\rangle}^{+}$ relation.

We turn now to the main task of obtaining a fusion formula for the transfer matrix. This involves a generalization of the trick employed in [6]. Substituting into the definition (4.5) of $\tilde{t}(u)$ the expressions (3.9), (4.8) for $K_{\langle 12\rangle}^{+}(u)$ and $\mathcal{T}_{\langle 12\rangle}^{-}(u)$, respectively, we obtain

$$\begin{aligned}\tilde{t}(u) = \mathrm{tr}_{12}\{&\tilde{P}_{21}^{+} K_2^{+}(u+\rho) M_2^{-1} R_{12}(-2u-3\rho)\\ &\times M_2 K_1^{+}(u)\mathcal{T}_1^{-}(u) R_{21}(2u+\rho)\mathcal{T}_2^{-}(u+\rho)\}.\end{aligned} \tag{4.11}$$

Using the identity $\tilde{P}_{21}^{+} = 1 - \tilde{P}_{21}^{-}$, we obtain the difference of two terms:

$$\begin{aligned}\tilde{t}(u) = \mathrm{tr}_{12}\{&K_2^{+}(u+\rho) M_1 R_{12}(-2u-3\rho)\\ &\times M_1^{-1} K_1^{+}(u)\mathcal{T}_1^{-}(u) R_{21}(2u+\rho)\mathcal{T}_2^{-}(u+\rho)\}\\ &-\mathrm{tr}_{12}\{\tilde{P}_{21}^{-} K_2^{+}(u+\rho) M_2^{-1} R_{12}(-2u-3\rho)\\ &\times M_2 K_1^{+}(u)\mathcal{T}_1^{-}(u) R_{21}(2u+\rho)\mathcal{T}_2^{-}(u+\rho)\}.\end{aligned}$$

The first term can be expressed as a product of two transfer matrices, while the second term can be expressed as a product of quantum determinants. Indeed, the first term can be cast as

$$\begin{aligned}\mathrm{tr}_{12}\{&\left[K_2^{+}(u+\rho) K_1^{+}(u)^{t_1} M_1^{-1} R_{12}(-2u-3\rho)^{t_1} M_1\right]^{t_1}\\ &\times \left[\mathcal{T}_1^{-}(u) R_{21}(2u+\rho)\mathcal{T}_2^{-}(u+\rho)\right]\}\\ &= \mathrm{tr}_{12}\{K_2^{+}(u+\rho) K_1^{+}(u)^{t_1} M_1^{-1} R_{12}(-2u-3\rho)^{t_1}\\ &\times M_1 \left[\mathcal{T}_1^{-}(u) R_{21}(2u+\rho)\mathcal{T}_2^{-}(u+\rho)\right]^{t_1}\}\\ &= \mathrm{tr}_{12}\{K_2^{+}(u+\rho) K_1^{+}(u)^{t_1} M_1^{-1} R_{12}(-2u-3\rho)^{t_1}\\ &\times M_1 R_{21}(2u+\rho)^{t_1}\mathcal{T}_1^{-}(u)^{t_1}\mathcal{T}_2^{-}(u+\rho)\}.\end{aligned}$$

Using the fact

$$M_1^{-1} R_{12}(-2u-3\rho)^{t_1} M_1 R_{21}(2u+\rho)^{t_1} = \zeta(2u+2\rho) \tag{4.12}$$

we conclude that the first term is equal to

$$\zeta(2u+2\rho)t(u)t(u+\rho).$$

The second term can be written as

$$\mathrm{tr}_{12}\{\tilde{P}^{-}_{21}K^{+}_{2}(u+\rho)M_{2}^{-1}R_{12}(-2u-3\rho)\\ \times M_{2}K^{+}_{1}(u)\tilde{P}^{-}_{12}\mathcal{T}^{-}_{1}(u)R_{21}(2u+\rho)\mathcal{T}^{-}_{2}(u+\rho)\}. \tag{4.13}$$

Expressing $\tilde{P}^{-}_{21}$ in terms of $\tilde{P}^{-}_{12}$ with the help of the identity

$$\tilde{P}^{-}_{21} = V_1 V_2 \tilde{P}^{-}_{12} V_1 V_2 \tag{4.14}$$

and using the property (2.9) of the projector $\tilde{P}^{-}_{12}$, we conclude that the second term is equal to a product of quantum determinants

$$\Delta\left\{K^{+}(u)\right\}\Delta\left\{\mathcal{T}^{-}(u)\right\}$$

where

$$\Delta\left\{K^{+}(u)\right\} = \mathrm{tr}_{12}\left\{\tilde{P}^{-}_{12}V_1V_2K^{+}_{2}(u+\rho)M_{2}^{-1}R_{12}(-2u-3\rho)M_{2}K^{+}_{1}(u)\right\} \tag{4.15}$$

$$\Delta\left\{\mathcal{T}^{-}(u)\right\} = \mathrm{tr}_{12}\left\{\tilde{P}^{-}_{12}\mathcal{T}^{-}_{1}(u)R_{21}(2u+\rho)\mathcal{T}^{-}_{2}(u+\rho)V_1V_2\right\}. \tag{4.16}$$

In short, we have the following fusion formula for the transfer matrix

$$\tilde{t}(u) = \zeta(2u+2\rho)t(u)t(u+\rho) - \Delta\left\{K^{+}(u)\right\}\Delta\left\{\mathcal{T}^{-}(u)\right\}. \tag{4.17}$$

This formula, which expresses the fused transfer matrix in terms of quantities related to the original system, is the main result of this paper. The next section is devoted to evaluating the quantum determinants that appear in this formula.

5. Evaluation of quantum determinants

Recalling the expression (4.2) for $\mathcal{T}^{-}(u)$, one can show that $\Delta\{\mathcal{T}^{-}(u)\}$ factors into a product of quantum determinants,

$$\Delta\left\{\mathcal{T}^{-}(u)\right\} = \delta\left\{T(u)\right\}\Delta\left\{K^{-}(u)\right\}\delta\left\{\hat{T}(u)\right\} \tag{5.1}$$

where

$$\begin{aligned}\delta\left\{T(u)\right\} &= \mathrm{tr}_{12}\left\{\tilde{P}^{-}_{12}T_1(u)T_2(u+\rho)\right\}\\ \delta\left\{\hat{T}(u)\right\} &= \mathrm{tr}_{12}\left\{\tilde{P}^{-}_{12}\hat{T}_2(u)\hat{T}_1(u+\rho)\right\}\end{aligned} \tag{5.2}$$

$$\Delta\left\{K^{-}(u)\right\} = \mathrm{tr}_{12}\left\{\tilde{P}^{-}_{12}K^{-}_{1}(u)R_{21}(2u+\rho)K^{-}_{2}(u+\rho)V_1V_2\right\}. \tag{5.3}$$

Let us now assume that the monodromy matrix $T(u)$ is given by the following product of R matrices

$$T_1(u) = R_{1,N}(u)R_{1,N-1}(u)\dots R_{1,1}(u) \tag{5.4}$$

and similarly

$$\hat{T}_1(u) = R_{1,1}(u)\ldots R_{N-1,1}(u)R_{N,1}(u). \tag{5.5}$$

With the help of the identity

$$\tilde{P}^-_{12} R_{1,m}(u) R_{2,m}(u+\rho) \tilde{P}^-_{12} = \zeta(u+\rho) \tilde{P}^-_{12} \qquad m = 1,2,\ldots,N, \tag{5.6}$$

(which can be proved using the unitarity and crossing-symmetry properties of $R(u)$, and the expression (2.7) for $\tilde{P}^-_{12}$), it follows that

$$\delta\left\{T(u)\right\} = \delta\left\{\hat{T}(u)\right\} = \zeta(u+\rho)^N. \tag{5.7}$$

In particular, we see explicitly that all the quantum determinants are c-numbers. It follows from the fusion formula (4.17) and the commutativity (4.9) that also

$$\left[\tilde{t}(u), \tilde{t}(v)\right] = 0. \tag{5.8}$$

Without further information about $K^{\mp}(u)$, the corresponding quantum determinants $\Delta\{K^{\mp}(u)\}$ cannot be evaluated. Here we consider the particular case [13, 14]

$$K^-(u) = 1 \qquad K^+(u) = M. \tag{5.9}$$

These are solutions of the reflection–factorization relations (3.1) and (3.2) provided that the R matrix satisfies

$$\left[\check{R}_{12}(u), \check{R}_{12}(v)\right] = 0 \tag{5.10}$$

where $\check{R} = \mathcal{P}R$. Jimbo has observed [12] that there is a large class of trigonometric R matrices for which this relation is valid. The corresponding open chains have [13, 14] quantum algebra symmetry. For these K matrices, we have

$$\begin{aligned}
\Delta\left\{K^-(u)\right\} &= \mathrm{tr}_{12}\left\{\tilde{P}^-_{12} V_1 V_2 R_{12}(2u+\rho)\right\} \\
\Delta\left\{K^+(u)\right\} &= \mathrm{tr}_{12}\left\{\tilde{P}^-_{12} V_1^{t_1} V_2^{t_2} R_{12}(-2u-3\rho)\right\}.
\end{aligned} \tag{5.11}$$

These expressions can be further simplified with the help of some new identities. The degeneration of the relation (5.10) at $v = -\rho$ yields

$$R_{12}(u)\tilde{P}^-_{21} = \tilde{P}^-_{12} R_{21}(u) \tag{5.12}$$

which implies

$$R_{12}(u) V_1 V_2 \tilde{P}^-_{12} = \tilde{P}^-_{12} V_1 V_2 R_{12}(u) \tag{5.13}$$

and therefore

$$\tilde{P}^-_{12} V_1 V_2 R_{12}(u) \tilde{P}^+_{12} = 0. \tag{5.14}$$

One can now show that

$$\tilde{P}_{12}^{-} V_1 V_2 R_{12}(u) \tilde{P}_{12}^{-} = g(u) \tilde{P}_{12}^{-} = \tilde{P}_{12}^{-} V_1^{t_1} V_2^{t_2} R_{12}(u) \tilde{P}_{12}^{-} \tag{5.15}$$

where the scalar function

$$g(u) = \mathrm{tr}_{12}\, R_{12}(u) V_1 V_2 \tilde{P}_{12}^{-} \tag{5.16}$$

is related to the function $\zeta(u)$ which appears in the unitarity relation (2.3) by

$$\zeta(u) = g(u) g(-u). \tag{5.17}$$

It should be emphasized that the decomposition (5.17) is not unique. We conclude that

$$\Delta\left\{K^-(u)\right\} = g(2u+\rho) \qquad \Delta\left\{K^+(u)\right\} = g(-2u-3\rho). \tag{5.18}$$

Remarkably, for the case of quantum algebra symmetry, all the quantum determinants in the fusion formula (4.17) are expressed entirely in terms of the known function $g(u)$.

6. Discussion

We have formulated a fusion procedure for open chains with PT-invariant R matrices. We have worked out in detail the example of a transfer matrix for which the auxiliary space is fused and the quantum space is not fused. These results can be directly applied [15] to exactly solving the large class of models with quantum algebra symmetry which we previously found [13, 14]. In order to construct and solve chains with spins in higher-dimensional representations, one must also fuse in the quantum space. This can be achieved simply by iterating the fusion procedure. (In the case of $A_1^{(1)}$, see [6].) In this way, one can generate an even larger class of integrable quantum-algebra-invariant chains.

The formulation of quantum current algebra of [16] bears a striking resemblance to that of integrable open chains with quantum algebra symmetry. The fusion results presented here should also be relevant for quantum current algebras in higher-dimensional representations.

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